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Some Basic Lommel Polynomials*

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The basic Lommel polynomials associated to the ${}_1\varphi_1$ q-Bessel function and the Jackson q-Bessel functions are considered as orthogonal polynomials in q^ν, where ν is the order of the corresponding basic Bessel functions. The corresponding

this measure is N-extremal. Some results on the zeros of the basic Bessel functions, both as functions of the order and of the argument are obtained. Precise asymptotic behaviour of the zeros of the ${}_1\varphi_1$ q-Bessel function is obtained.

INTRODUCTION

An orthogonality measure for orthogonal polynomials can be derived if the asymptotic behaviour of the orthogonal polynomials is known. In case the coefficients in the three-term recurrence relation are unbounded, the orthogonality measure is supported on an infinite interval. In this case it might happen that the corresponding moment problem is indeterminate, i.e., there is more than one orthogonality measure for the orthogonal polynomials. This problem goes back to Stieltjes's famous memoir "Recherches sur les fractions continues" from 1894–1895. Here we consider orthogonal polynomials arising from basic analogues of the Bessel function which are defined by a three-term recurrence relation in which the coefficients are exponentially increasing. It turns out that techniques developed by Chihara and Maki in the 1960s are useful in the study of these orthogonal polynomials.

Let us start with describing the situation for the Lommel polynomials as orthogonal polynomials, both as polynomial in the argument or as

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polynomial in the order. The Lommel polynomials, introduced by Lommel in 1871, are polynomials related to the Bessel function

$$J_\nu(z)=\sum_{k=0}^{\infty}\frac{(-1)^k\,(z/2)^{2k+\nu}}{k!\,\Gamma(\nu+k+1)}=\frac{(z/2)^\nu}{\Gamma(\nu+1)}\,{}_0F_1(-;\nu+1;z^2/4), \tag{1.1}$$

of order ν and argument z. The Bessel function is a very well-understood special function; see Watson's treatise [23]. A recurrence relation for the Bessel function is, cf. [23, Sect. 3.2(1)],

$$J_{\nu+1}(z)=\frac{2\nu}{z}\,J_\nu(z)-J_{\nu-1}(z). \tag{1.2}$$

Iterating this relation shows that $J_{\nu+m}(z)$ can be expressed in terms of $J_\nu(z)$ and $J_{\nu-1}(z)$, such that the coefficients are polynomials in z^{-1},

$$J_{\nu+m}(z)=h_{m,\nu}(z^{-1})\,J_\nu(z)-h_{m-1,\nu-1}(z^{-1})\,J_{\nu-1}(z),$$

where $h_{m,\nu}(z)$ are the Lommel polynomials, see [23, Chap. 9].

The Lommel polynomials satisfy

$$h_{m+1,\nu}(z)=2z(m+\nu)\,h_{m,\nu}(z)-h_{m-1,\nu}(z), \quad h_{-1,\nu}(z)=0, \quad h_{0,\nu}(z)=1. \tag{1.3}$$

Favard's theorem, see, e.g., [5, Chap. II, Theorem 6.4], shows that for $\nu>0$ the Lommel polynomials are orthogonal with respect to a positive measure on the real line. The explicit orthogonality relations for the Lommel polynomials have been determined in papers by Schwartz, Dickinson, Dickinson, Pollak and Wannier, and Goldberg in 1940–1965 in terms of the zeros of the Bessel function of order $\nu-1$; see the references in [9, 16, 18]. An ingredient for the derivation of the orthogonality relations is Hurwitz's asymptotic formula, see [23, 9.65(1)],

$$\frac{(2z)^{1-\nu-m}\,h_{m,\nu}(z)}{\Gamma(\nu+m)}\to J_{\nu-1}\left(\frac{1}{z}\right), \qquad m\to\infty. \tag{1.4}$$

Putting $r_{m,z}(\nu)=h_{m,\nu}(z)$ we see from (1.3) that $r_{m,z}(\nu)$ also satisfies a three-term recurrence relation to which Favard's theorem is applicable. Hence, for each real non-zero z there is a positive measure on the real line for which the polynomials $r_{m,z}(\nu)$, i.e., the Lommel polynomials considered as polynomials of the order ν, are orthogonal. This measure has unbounded support. It has been determined explicitly by Maki [18] in 1968 by determining the Stieltjes transform of the orthogonality measure in terms of Bessel functions using Hurwitz's asymptotic formula (1.4). Maki's results give information on the zeros of $\nu \mapsto J_\nu(z)$ for real z. Specifically, the zeros are real and simple and form a denumerable set. See also Coulomb [6] for analytic derivations of related results.

In this paper we consider a similar situation for basic analogues of the Bessel function, namely for the ${}_1\varphi_1$ q-Bessel functions and for Jackson's q-Bessel functions. The ${}_1\varphi_1$ q-Bessel functions are also known as the Hahn–Exton q-Bessel functions, and here the orthogonality for the q-Lommel polynomials in q^ν seems the most natural. In Section 2 we recall the results needed on the ${}_1\varphi_1$ q-Bessel function and on the associated q-Lommel polynomials. In Section 3 we state orthogonality relations for the q-Lommel polynomials as polynomials in $x = q^\nu$. Here indeterminate moment problems arise depending on the value of the argument, and the constructed orthogonality measure is N-extremal in the indeterminate case. Using a duality between argument and order we obtain results on the zeros of the ${}_1\varphi_1$ q-Bessel function considered as function of the argument, and in particular we find that the positive zeros tend to infinity as $q^{-(1/2)k}$. Finally, in Section 4 we give related results for the q-Lommel polynomials associated to Jackson's q-Bessel function. In this case the moment problem is determinate.

The main tools are Maki's general theorem [18, Theorem 4.9] and Chihara's theory of chain sequences related to orthogonal polynomials, see [3–5]. In all cases the coefficients in the three-term recurrence relation are exponentially increasing, and in some cases the moment problem is indeterminate as follows from the asymptotic behaviour. The orthogonal polynomials considered all fit into the class of q-like orthogonal polynomials studied by Van Assche [21], so that the asymptotic behaviour of the rescaled polynomials is available. However, this doesn't suffice for our purposes.

The Lommel polynomials also have an interpretation on the group of plane motions arising from the well-known interpretation of Bessel functions on this group, see Feinsilver [7]. The ${}_1\varphi_1$ q-Bessel functions also have a natural interpretation on the quantum group of plane motions, see [13, 20], and we might ask whether Feinsilver's interpretation goes through in the quantum group setting. The quantum group interpretation also holds for a two-parameter q-Bessel function extending the ${}_1\varphi_1$ q-Bessel functions, see [14]. These q-Bessel functions have also been studied by Ismail *et al.* [11], and we might expect that the appropriate generalisations of the results of this paper exist in some sense for this more general q-Bessel function.

Notation. The basic convention throughout the paper is $0 < q < 1$. The notation for basic (or q-)hypergeometric series follows Gasper and Rahman [8].

2. THE ${}_1\varphi_1$ q-BESSEL FUNCTION AND q-LOMMEL POLYNOMIALS

In this section we recall the basic Lommel polynomials associated with the ${}_1\varphi_1$ q-Bessel function studied in [15, 16]. This q-Bessel function is also known as the Hahn–Exton q-Bessel function, but its history actually goes back to Jackson's 1904 paper [12, Sect. 6], where this function appears in rather awkward notation.

The ${}_1\varphi_1$ q-Bessel function is defined by

$$
\begin{aligned}
J_\nu(w;q) &= \sum_{k=0}^{\infty} \frac{(q^{\nu+k+1};q)_\infty}{(q;q)_\infty} \frac{(-1)^k q^{1/2k(k+1)} w^{\nu+2k}}{(q,q)_k} \\
&= w^\nu \frac{(q^{\nu+1};q)_\infty}{(q;q)_\infty} \, {}_1\varphi_1(0;q^{\nu+1};q,qw^2) \\
&= w^\nu \frac{(qw^2;q)_\infty}{(q;q)_\infty} \, {}_1\varphi_1(0;qw^2;q,q^{\nu+1}),
\end{aligned} \tag{2.1}
$$

where the last equality follows from $(y;q)_\infty \, {}_1\varphi_1(0;y;q,x) = (x;q)_\infty \, {}_1\varphi_1(0;x;q,y)$, see [17, (2.3)]. So $\mathscr{J}_\nu(w;q) = w^{-\nu} J_\nu(w;q)$ is an entire function in both w and $x = q^\nu$. This q-Bessel function satisfies the recurrence relation, cf. (1.2),

$$
J_{\nu+1}(w;q) + J_{\nu-1}(w;q) = \left(w + \frac{1-q^\nu}{w} \right) J_\nu(w;q). \tag{2.2}
$$

See [15] and references given there.

The q-Lommel polynomials $h_{m,\nu}(w;q)$ that follow from iterating (2.2) satisfy

$$
(w^{-1} + w(1-q^{\nu+m}))\, h_{m,\nu}(w;q) = h_{m+1,\nu}(w;q) + h_{m-1,\nu}(w;q), \tag{2.3}
$$

with initial conditions $h_{0,\nu}(w;q) = 1$, $h_{1,\nu}(x;q) = w^{-1} + (1-q^\nu)\, w$, see [15, Proposition 4.3]. In [16] these polynomials have been considered as orthogonal Laurent polynomials in w and the orthogonality measure has been determined explicitly in terms of the ${}_1\varphi_1$ q-Bessel function of order $\nu - 1$. The analogue of Hurwitz's formula (1.4) is slightly more complicated, see [15, 16];

$$
\lim_{m\to\infty} w^{-m} h_{m,\nu}(w;q) = \frac{(q;q)_\infty}{(w^{-2};q)_\infty} \mathscr{J}_{\nu-1}(w^{-1};q), \qquad |w| > 1. \tag{2.4}
$$

For $|w| < 1$ the asymptotic behaviour can be expressed in terms of a function closely related to the ${}_1\varphi_1$ q-Bessel function, see [16].

It is clear from (2.3) that we can consider $h_{m,\nu}(w;q)$ as a polynomial of degree m in $x=q^\nu$. Put $P_m(q^\nu;w;q)=h_{m,\nu}(w;q)$. Then we can rewrite (2.3) as

$$\begin{aligned} xP_m(x;w;q) = &-w^{-1}q^{-m}P_{m+1}(x;w;q)+q^{-m}(1+w^{-2})\,P_m(x;w;q)\\ &-w^{-1}q^{-m}P_{m-1}(x;w;q), \end{aligned} \tag{2.5}$$

with initial condition $P_0(x;w;q)=1$, $P_1(x;w;q)=w+w^{-1}-wx$. Favard's theorem, see, e.g., [5, Chap. II, Theorem 6.4], shows that $P_m(\cdot;w;q)$ are orthogonal polynomials on the real line for some positive Borel measure for $w\in\mathbb{R}\backslash\{0\}$. From (2.5) we obtain $P_m(x;-w;q)=(-1)^m P_m(x;w;q)$, so that we may restrict ourselves to positive w. From (2.5) we also get $P_m(w^{-2}x;w;q)=P_m(x;w^{-1};q)$, so that we may restrict ourselves to $w\geqslant 1$.

For the sake of completeness we recall the following explicit expression, see [15, Sect. 4],

$$P_m(x;w;q)=\sum_{j=0}^{m} x^j w^m q^{jm}\frac{(q^{-m};q)_j}{(q;q)_j}\,{}_2\varphi_1\left(\begin{matrix} q^{j-m},\ q^{j+1} \\ q^{-m} \end{matrix};q,\frac{q^{-j}}{w^2}\right). \tag{2.6}$$

Hence, $P_m(0;w;q)=\sum_{k=0}^m w^{m-2k}=U_m(1/2(w+w^{-1}))$, where U_m denotes a Chebyshev polynomial of the second kind. This also follows from (2.5), see [16, Sect. 7]. Now we get the analogue of (2.4),

$$\lim_{m\to\infty} w^{-m}P_m(x;w;q)=\frac{1}{(1-w^{-2})}\,{}_1\varphi_1(0,qw^{-2};q,x), \qquad |w|>1, \tag{2.7}$$

uniformly in x on compact subsets of $\mathbb{C}$ as can be derived from the explicit expression (2.6) and dominated convergence.

The monic polynomial $p_m(x)=(-w)^{-m}q^{-(1/2)m(m-1)}P_m(x;w;q)$ satisfies

$$\begin{aligned} &xp_m(x)=p_{m+1}(x)+\alpha_m p_m(x)+\beta_m p_{m-1}(x),\\ &\alpha_m=q^{-m}(1+w^{-2}), \qquad \beta_m=w^{-2}q^{1-2m}, \end{aligned} \tag{2.8}$$

with initial condition $p_0(x)=1$, $p_1(x)=x-w^{-2}-1$. The corresponding orthonormal polynomials $r_m(x)=|w|^m q^{(1/2)m^2}p_m(x)=(-\operatorname{sgn} w)^m q^{(1/2)m} P_m(x;w;q)$ satisfy

$$\begin{aligned} &xr_m(x)=a_{m+1}r_{m+1}(x)+b_m r_m(x)+a_m r_{m-1}(x),\\ &a_m=|w|^{-1}q^{1/2-m}, \qquad b_m=q^{-m}(1+w^{-2}), \end{aligned} \tag{2.9}$$

with initial condition $r_0(x)=1$, $r_1(x)=|w|\,x-|w|-|w|^{-1}$. Note that the coefficients in (2.9) are positive and exponentially increasing.

Observe that (2.8) is closely related to the three-term recurrence relation for the Al-Salam and Carlitz polynomials $V_m^{(a)}(x)$, see Al-Salam and Carlitz [1],

$$xV_m^{(a)}(x) = V_{m+1}^{(a)}(x) + (1+a)\,q^{-m}V_m^{(a)}(x) + aq^{1-2m}(1-q^m)\,V_{m-1}^{(a)}(x), \tag{2.10}$$

by taking $a = w^{-2}$ and we may expect some similarities in the analysis. However, the difference in the coefficients of the $(m-1)$ degree polynomial in (2.8) and (2.10) is unbounded as $m \to \infty$.

3. ORTHOGONALITY AND ZEROS

In this section we investigate the orthogonality properties for the q-Lommel polynomials of (2.5) and zeros of the ${}_1\varphi_1$ q-Bessel function both as a function of the argument and of the order. We establish the orthogonality measure for the orthogonal polynomials $P_m(\cdot; w; q)$ in case the corresponding moment problem is determinate. For the indeterminate cases we present an N-extremal measure except for the case $w = 1$. It turns out that the zeros of the ${}_1\varphi_1$ q-Bessel function (as a function of the argument) describe the point masses of the measures. Using this information we establish precise growth behaviour of the zeros of the ${}_1\varphi_1$ q-Bessel of order greater than -1.

LEMMA 3.1. *Let $w \geqslant 1$, and $P_m(\cdot; w; q)$ be defined by* (2.5).

(i) *The Hamburger moment problem for the orthogonal polynomials $P_m(\cdot; w; q)$ is determinate for $w \geqslant q^{-1/2}$ and indeterminate for $1 \leqslant w < q^{-1/2}$*

(ii) *The true interval of orthogonality, i.e., the smallest closed interval containing all the zeros of all the polynomials $P_m(\cdot; w; q)$, is contained in* $[0, \infty)$.

Part (i) of Lemma 3.1 corresponds nicely to the similar statement for the Al-Salam and Carlitz polynomials in Chihara [4, Sect. 5(B)].

Remark. The three-term recurrence relation (2.5) can be viewed as a three-term recurrence relation as occurring in birth and death processes with values $\mu_m = q^{-m}$, $\lambda_m = w^{-2}q^{-m}$ after putting $F_m(x) = (q/w)^m\,P_m(x; w; q)$, where we refer to [10] for the notation as well as an introduction to this subject. This implies Lemma 3.1(ii) by [10, Sect. 2]. For the birth and death processes another set of orthogonal polynomials, with $\mu_0 = 0$ instead of $\mu_0 = 1$, is also of interest. I thank the referee for pointing this out.

Proof. To prove (i) we observe that by (2.7) we have

$$P_m(x; w; q) \approx \frac{w^m}{1-w^{-2}}\, {}_1\varphi_1(0; qw^{-2}; q, x), \qquad w > 1.$$

Pick any $x \in \mathbb{C} \setminus \mathbb{R}$ for which the function ${}_1\varphi_1(0; qw^{-2}; q, x)$ is non-zero. Then we find that $r_m(x)$ behaves like $(wq^{1/2})^m$, so that $\sum_{m=0}^{\infty} |r_m(x)|^2$ diverges if and only if $w \geqslant q^{-1/2}$. Now (i) follows from [2, Theorem 1.1, p. 503; 19, Theorem 2.9, p. 50], apart from the case $w=1$. For $w=1$ we have $P_m(x; 1; q) = \mathcal{O}(m)$, see [16, Section 7] or apply Darboux's method to [15, (4.22)] and $r_m(x)$ behaves like $mq^{(1/2)\,m}$, so that $\sum_{m=0}^{\infty} |r_m(x)|^2$ converges for $w=1$.

For (ii) we use Theorem 1 of Chihara [3], see also [5], stating that a necessary and sufficient condition for the true interval of orthogonality being contained in $[c, \infty)$ is $\alpha_n > c$, $n \in \mathbb{Z}_+$, and $\{\beta_m/((\alpha_m - c)(\alpha_{m-1} - c))\}_{m=1}^{\infty}$ a chain sequence. Recall that $\{h_m\}_{m=1}^{\infty}$ is a chain sequence if there exists a parameter sequence $\{g_m\}_{m=0}^{\infty}$, with $0 \leqslant g_0 < 1$ and $0 < g_m < 1$ for $m \geqslant 1$, such that $h_m = (1 - g_{m-1})\, g_m$, see [5, Chap. III]. Take $c=0$. The first condition is trivially satisfied and

$$\frac{\beta_m}{\alpha_m \alpha_{m-1}} = \frac{w^{-2}}{(1+w^{-2})^2} \in \left(0, \frac{1}{4}\right],$$

which is a chain sequence with parameter sequence $g_m = (1+w^{-2})^{-1}$. ∎

Remark. 3.2. In case $1 \leqslant w < q^{-1/2}$ the Hamburger moment problem is indeterminate from Lemma 3.1(i) and by part (ii) the Stieltjes moment problem has a solution, since there is always an orthogonality measure supported on the true interval of orthogonality; see [5, 19]. Since the chain sequence $\{\beta_m/(\alpha_m \alpha_{m-1})\}_{m=1}^{\infty}$ doesn't determine its parameter sequence uniquely, it follows from [4, Theorem 1] that for $1 \leqslant w < q^{-1/2}$ the Stieltjes moment problem is also indeterminate. Indeed, the parameter sequence $g_m = (1+w^{-2})^{-1}$ is the maximal parameter sequence, see [5, Chap. III].

Remark 3.3. Following [5] we denote the true interval of orthogonality by $[\xi_1, \eta_1]$, where ξ_1, respectively η_1, is the limit of the first, respectively last, zero of $p_m(x)$ as $m \to \infty$. From [5, Corollary 2, p. 109] we obtain $\eta_1 = \infty$. More generally, let ξ_i denote the limit of the ith zero of $p_m(x)$ as $m \to \infty$. Then $\xi_i < \xi_{i+1}$ and it follows from [5, (3.7), (3.8), p. 119] that $\Xi = \{\xi_1, \xi_2, ...\}$ has no point of accumulation and that there exists a discrete orthogonality measure with Ξ as its spectrum.

LEMMA 3.4. *For the true interval of orthogonality* $[\xi_1, \infty)$ *we have* $1 + w^{-2} > \xi_1 > 0$ *and for* $w \geqslant q^{-1/2}$ *we have* $\xi_1 \geqslant \sigma(w) = 1 + w^{-2} - w^{-1}(q^{1/2} + q^{-1/2})$.

Proof. From Chihara's characterisation as used in the proof of Lemma 3.1, we see that $\inf \alpha_m = 1 + w^{-2}$ is an upper bound for ξ_1.

A straightforward calculation shows that for the coefficients in (2.9) we have

$$\begin{aligned} b_m - a_m - a_{m+1} &= q^{-m}\sigma(w), \qquad && m \geqslant 1, \\ b_0 - a_1 &= \sigma(w) + |w|^{-1} q^{1/2} \qquad && m = 0. \end{aligned} \tag{3.1}$$

This is positive for $w > q^{-1/2}$ and zero for $w = q^{-1/2}$. On the other hand, the zeros of $r_m(x)$ are the eigenvalues of a truncated Jacobi matrix of size m, see, e.g., Van Assche [22, p. 226]. This symmetric matrix is tridiagonal with $b_0, ..., b_{m-1}$ on the diagonal and $a_1, ..., a_{m-1}$ and the super- and subdiagonal. Hence, Gershgorin's theorem implies that the eigenvalues, hence the zeros of $r_m(x)$, are contained in the disks $\{x\colon |x - b_k| \leqslant a_k + a_{k+1}\}$, $k = 1, ..., m-2$, and $\{x\colon |x - b_0| \leqslant a_1\}$ and $\{x\colon |x - b_{m-1}| \leqslant a_{m-1}\}$. Since zeros of orthogonal polynomials are real, we get $\xi_1 \geqslant \sigma(w) \geqslant 0$ for $w \geqslant q^{-1/2}$.

For $1 \leqslant w < q^{-1/2}$ we can apply [4, Theorem 4], stating that $\xi_1 = 0$ if and only if the Stieltjes moment problem is determinate. As remarked in Remark 3.2 this is not the case, so $\xi_1 > 0$.

It remains to consider $w = q^{-1/2}$. Since $0 < q < 1$ we can find $\varepsilon > 0$ such that $\alpha_n > \varepsilon(1+q)$ and

$$\frac{\beta_m}{(\alpha_m - \varepsilon(1+q))(\alpha_{m-1} - \varepsilon(1+q))} = \frac{q}{(1+q)^2} \frac{1}{(1-\varepsilon q^m)(1-\varepsilon q^{m-1})} \leqslant \frac{1}{4},$$

implying that the left hand side is a chain sequence by Wall's comparison test, see [5, Theorem 5.7, p. 97]. Using [3, Theorem 1] as in Lemma 3.1, we see that $\xi_1 \geqslant \varepsilon(1+q)$. ▮

Remark. (i) The fact that (3.1) is non-negative for $w \geqslant q^{-1/2}$ implies determinacy of the Hamburger moment problem by Berezanskiĭ [2, Corollary, p. 506].

(ii) To see that $\xi_1 > 0$ for $w > 1$ we can also apply Chihara [3, Lemma 6], which gives a lower bound for ξ_1 in terms of $\inf \alpha_m = (1 + w^{-2})$, $\sup \beta_m/(\alpha_m \alpha_{m-1}) = w^{-2}/(1 + w^{-2})$ and the infimum of the difference between the maximal and minimal parameter sequence for the chain sequence $\beta_m/(\alpha_m \alpha_{m-1})$. Since the maximal parameter sequence is the constant sequence $(1 + w^{-2})^{-1} > 1/2$ for $w > 1$ and the minimal parameter sequence is less than 1/2 the infimum is positive, cf. [5, Corollary, p. 102], and then Chihara's estimate gives $\xi_1 > 0$. The explicit estimate is

$$\xi_1 \geqslant (1+w^{-2})\left(1-\frac{4w^{-1}}{\sqrt{1+14w^{-2}+w^{-4}}}\right), \tag{3.2}$$

which is positive for $w>1$.

Next we calculate the associated polynomials $P_m^{(1)}(x;w;q)$;

$$\begin{aligned} xP_m^{(1)}(x;w;q) = &-w^{-1}q^{-m-1}P_{m+1}^{(1)}(x;w;q) \\ &+q^{-m-1}(1+w^{-2})\,P_m^{(1)}(x;w;q) \\ &-w^{-1}q^{-m-1}P_{m-1}^{(1)}(x;w;q), \end{aligned} \tag{3.3}$$

with initial conditions $P_0^{(1)}(x;w;q)=1$, $P_1^{(1)}(x;w;q)=w+w^{-1}-qwx$. Comparing (3.3) with (2.5) yields $P_m^{(1)}(x;w;q)=P_m(qx;w;q)$. Hence, the associated monic polynomials are $p_m^{(1)}(x)=q^{-m}p_m(qx)$. The polynomials $p_m(x)$ and $p_m^{(1)}(x)$ are the denominator and numerator polynomial of the continued fraction $K(x;w;q)$ giving the Stieltjes transform $K(x;w;q)=\int_{-\infty}^{\infty} dm(t)/(x-t)$ of an orthogonality measure for the orthogonal polynomials, see Shohat and Tamarkin [19, Sect. 12] or [5, 18]. So, if the continued fraction converges, it is given by

$$\begin{aligned} K(x;w;q) &= \lim_{m\to\infty} \frac{p_{m-1}^{(1)}(x)}{p_m(x)} = \lim_{m\to\infty} q^{1-m}\frac{p_{m-1}(qx)}{p_m(x)} \\ &= \lim_{m\to\infty} -w\,\frac{P_{m-1}(qx;w;q)}{P_m(x;w;q)}. \end{aligned}$$

We can calculate $K(x;w;q)$ for $w>1$ using (2.7). Hence,

$$K(x;w;q) = -\frac{{}_1\varphi_1(0;qw^{-2};q,qx)}{{}_1\varphi_1(0;qw^{-2};q,x)}, \qquad w>1. \tag{3.4}$$

Note that $K(x;w;q)$ is a meromorphic function in x. The next lemma is an application of a theorem by Maki [18] generalizing the analysis for the orthogonality of the Lommel polynomials as polynomials in w, see also [3, Theorem 8]. Recall from Remark 3.3 that the ξ_i is the limit of the ith zero of p_m as $m\to\infty$.

LEMMA 3.5. *For $w>1$ the function $K(x;w;q)$ is meromorphic and it has a Mittag–Leffler expansion*

$$K(x;w;q)=\sum_{k=1}^{\infty}\frac{A_k}{x-\xi_k},\qquad A_k>0,\quad \sum_{k=1}^{\infty}A_k=1,$$

and $0<\xi_1<\xi_2<\cdots$ tending to infinity. Moreover, an orthogonality measure for the orthogonal polynomials $P_m(\cdot;w;q)$ is purely discrete with masses A_k at the points ξ_k. Moreover, if $w\geqslant q^{-1/2}$, then the orthogonality measure is unique.

Proof. Since obviously $\alpha_m\to\infty$ and $\limsup_{m\to\infty}\beta_m/(\alpha_m\alpha_{m-1})=w^{-2}/(1+w^{-2})^2<1/4$ for $w>1$, we can apply Theorem 4.9 of Maki [18] to find

$$K(x;w;q)=\sum_{k=1}^{\infty}\frac{A_k}{x-x_k},\qquad A_k>0,\quad \sum_{k=1}^{\infty}A_k=1,$$

for certain real points x_i satisfying $-\infty<x_1<x_2<\cdots$. Moreover, an orthogonality measure for the orthogonal polynomials is purely discrete with masses A_k at the points x_k. Since the spectrum of the orthogonality measure has to be unbounded, because the coefficients in (2.9) are unbounded, we get $x_k\to\infty$.

Next we observe that for $w>1$ by (2.7)

$$\lim_{m\to\infty}\frac{p_m(x)}{p^{(1)}_{m-1}(0)}=\lim_{m\to\infty}-\frac{w^{-m}P_m(x;w;q)}{w^{-(m-1)}U_{m-1}((1/2)(w+w^{-1}))}$$
$$=-{}_1\varphi_1(0;qw^{-2};q,x). \tag{3.5}$$

By Hurwitz's theorem on uniform convergence of analytic functions we conclude that the zeros of $x\mapsto {}_1\varphi_1(0;qw^{-2};q,x)$ consist of $\Xi=\{\xi_1,\xi_2,...\}$. In order to conclude that $x_i=\xi_i$ for $i\in\mathbb{N}$ we need to check that the numerator and denominator of $K(x;w;q)$ in (3.4) have no common zeros. Indeed, $x=0$ is no common zero, and if $x\neq 0$ and xq are zeros of ${}_1\varphi_1(0;qw^{-2};q,x)$, then by the second order q-difference equation for the ${}_1\varphi_1$-series, derivable from [8, Ex. 1.13], we also get xq^2 as a zero. Iterating the argument gives xq^k, $k\in\mathbb{Z}_+$, as a zero, so that by analyticity ${}_1\varphi_1(0;qw^{-2};q,x)$ would be identically zero, which is absurd.

For $w\geqslant q^{-1/2}$ the statements follow from Lemmas 3.1 and 3.4. ∎

The proof of the following theorem is much motivated by Ismail's investigation [9] of the zeros of the Jackson q-Bessel function.

THEOREM 3.6. *Let* $w > 1$.

(i) *The zeros of* $x \mapsto {}_1\varphi_1(0; qw^{-2}; q, x)$ *are real, simple, and form a denumerable set* $0 < \xi_1 < \xi_2 < \cdots$ *with* $\xi_k \to \infty$ *as* $k \to \infty$. *Moreover, the zeros of* $x \mapsto {}_1\varphi_1(0; qw^{-2}; q, x)$ *interlace with the zeros of* $x \mapsto {}_1\varphi_1(0; qw^{-2}; q, qx)$

(ii) *The q-Lommel polynomials defined by* (2.5) *satisfy the following orthogonality relations with respect to a positive measure*;

$$-\sum_{k=1}^{\infty} P_m(\xi_k; w; q)\, P_n(\xi_k; w; q) \frac{{}_1\varphi_1(0; qw^{-2}; q, q\xi_k)}{(\partial/\partial x)({}_1\varphi_1(0; qw^{-2}; q, x))|_{x=\xi_k}} = \delta_{nm} q^{-m}.$$

Moreover, for $w \geqslant q^{-1/2}$ *the measure is unique, and for* $1 < w < q^{-1/2}$ *the measure is a N-extremal solution of the indeterminate Hamburger moment problem.*

Recall, see [19, p. 57], that in case of an indeterminate moment problem the Stieltjes transform of the convex set of measures having the same moments can be parametrised by holomorphic functions ϕ in the upper half plane with non-positive imaginary part,

$$\int_{-\infty}^{\infty} \frac{d\mu_\phi(x)}{z-x} = \frac{A(z) - \phi(z)\, C(z)}{B(z) - \phi(z)\, D(z)},$$

where A, B, C, and D are entire functions. This is the Nevanlinna parametrisation. An N-extremal measure corresponds to ϕ being a constant $c \in \mathbb{R} \cup \{\infty\}$. The N-extremal measures, by a theorem of M. Riesz, characterise the measures for which the polynomials are dense in the corresponding weighted L^2-space; see [19, p. 62].

Proof. Combine Lemma 3.5 with (3.4) to see that

$$\sum_{k=1}^{\infty} \frac{A_k}{x - \xi_k} = -\frac{{}_1\varphi_1(0; qw^{-2}; q, qx)}{{}_1\varphi_1(0; qw^{-2}; q, x)} \tag{3.6}$$

with $0 < \xi_1 < \xi_2 < \cdots$ and $\xi_k \to \infty$. So the zeros of $x \mapsto {}_1\varphi_1(0; qw^{-2}; q, x)$ are positive, simple, and form a denumerable set $\{\xi_1, \xi_2, ...\}$, since the numerator and denominator have no common zeros.

From (3.6) we obtain

$$0 < A_k = -\frac{{}_1\varphi_1(0; qw^{-2}; q, q\xi_k)}{(\partial/\partial x)({}_1\varphi_1(0; qw^{-2}; q, x))|_{x=\xi_k}}.$$

Since the zeros are real and simple the denominator has opposite signs for k and $k+1$. Hence, ${}_1\varphi_1(0; qw^{-2}; q, q\xi_k)\, {}_1\varphi_1(0; qw^{-2}; q, q\xi_{k+1}) < 0$, or $x \mapsto {}_1\varphi_1(0; qw^{-2}; q, qx)$ has at least one zero in (ξ_k, ξ_{k+1}). To show that there is precisely one zero we differentiate (3.6) with respect to x. The left-hand side then gives $-\sum_{k=1}^{\infty}(A_k/(x-\xi_k)^2)$, which is negative on (ξ_k, ξ_{k+1}), so the right-hand side of (3.6) has at most one zero in (ξ_k, ξ_{k+1}), since it is differentiable in (ξ_k, ξ_{k+1}). This proves (i).

Because of Lemma 3.5 we obtain for the orthonormal polynomials $r_m(x)$ defined by (2.9) the orthogonality relations

$$\sum_{k=1}^{\infty} r_m(x_k)\, r_n(x_k)\, A_k = \delta_{nm} \sum_{k=1}^{\infty} A_k = \delta_{nm}.$$

Since $r_m(x) = (-1)^m q^{(1/2)m} P_m(x; w; q)$ we obtain (ii).

To show that for $1 < w < q^{-1/2}$ the measure is N-extremal we use Chihara [4, Sect. 4], showing that (3.5) implies that $-{}_1\varphi_1(0; qw^{-2}; q, z) = D(z) + B(z)$, or the measure corresponds to $\phi \equiv -1$, hence is N-extremal. ▮

Since the zeros of $x \mapsto {}_1\varphi_1(0; qw^{-2}; q, qx)$ are obviously $q^{-1}\xi_k$ the interlacing property of Theorem 3.6(i) just means that $q^{-1}\xi_k < \xi_{k+1}$. For the indeterminate case this also follows from the general Theorem 2.13 of [19, p. 60].

Note that for $1 < w < q^{-1/2}$ this is just one of the many possible inequivalent orthogonality measures corresponding to this indeterminate moment problem. Any other solution of the moment problem has a point in its spectrum smaller than ξ_1, see Chihara [4, Lemma 2]. It would be of interest to determine the entire functions $A(z)$, $B(z)$, $C(z)$, and $D(z)$ in the Nevanlinna parametrisation explicitly, so that all N-extremal measure can be found. Note that ${}_1\varphi_1(0; qw^{-2}; q, z)$ is an entire function of order zero.

Corollary 3.7. *Let* $w > 1$.

(i) *The zeros of* $\nu \mapsto \mathcal{J}_{\nu-1}(w^{-1}; q)$ *are real, simple, and form a denumerable set* $\cdots < \nu_{k+1} < \nu_k < \cdots < \nu_1 < \infty$ *with* $\nu_k \to -\infty$ *as* $k \to \infty$. *Moreover, the zeros of* $\nu \mapsto \mathcal{J}_{\nu-1}(w^{-1}; q)$ *interlace with the zeros of* $\nu \mapsto \mathcal{J}_{\nu}(w^{-1}; q)$.

(ii) *The q-Lommel polynomials defined by* (2.3) *satisfy the orthogonality relations*

$$-\ln q \sum_{k=1}^{\infty} h_{m,\nu_k}(w; q)\, h_{n,\nu_k}(w; q)\, q^{\nu_k} \frac{\mathcal{J}_{\nu_k}(w^{-1}; q)}{(\partial/\partial\nu)(\mathcal{J}_{\nu}(w^{-1}; q))|_{\nu=\nu_k-1}} = \delta_{nm} q^{-m}.$$

All masses are positive, and the orthogonality measure is uniquely determined for $w \geq q^{-1/2}$.

Proof. This is Theorem 3.6 with $x = q^{\nu}$ and $\xi_k = q^{\nu_k}$ using Lemma 3.4. ▮

We denote for $\nu > -1$ the ordered positive simple zeros of $w \mapsto \mathcal{J}_\nu(w; q)$ by $0 < j_1(\nu) < j_2(\nu) < \cdots$, $\lim_{k \to \infty} j_k(\nu) = \infty$, see [15, Theorem 3.7]. So we can rewrite Theorem 3.6(ii) or Corollary 3.7(ii) for $w = q^{-(1/2)\rho}$, $\rho > 0$, as

$$-2q \sum_{k=1}^{\infty} P_m(qj_k(\rho)^2; q^{-(1/2)\rho}; q)\, P_n(qj_k(\rho)^2; q^{-(1/2)\rho}; q) \times j_k(\rho) \frac{\mathcal{J}_\rho(q^{1/2} j_k(\rho); q)}{(\partial/\partial w)(\mathcal{J}_\rho(w; q))|_{w = j_k(\rho)}} = \delta_{nm} q^{-m}, \tag{3.7}$$

since $\xi_k = qj_k(\rho)^2$.

Corollary 3.8. (i) *For $\rho > 0$ we have $j_{k+1}(\rho) > q^{-1/2} j_k(\rho)$.*

(ii) *For $\rho \geqslant 1$ we have $j_1(\rho) \geqslant q^{-1/2} \sqrt{(1 - q^{(1/2)(\rho+1)})(1 - q^{(1/2)(\rho-1)})} \geqslant q^{-1/2}(1 - q^{(1/2)(\rho-1)})$.*

(iii) *For $\rho > 0$ we have*

$$q^{-1/2} \sqrt{1 + q^\rho} > j_1(\rho) \geqslant q^{-1/2} \sqrt{(1 + q^\rho)\left(1 - \frac{4q^{1/2\rho}}{\sqrt{1 + 14q^\rho + q^{2\rho}}}\right)}.$$

Note that (i) implies that the zeros of the ${}_1\varphi_1$ q-Bessel function are exponentially increasing. In Corollary 3.10 the estimate is complemented by also establishing an upper bound for the quotient of successive zeros. From (iii) we get that $j_1(\rho) \to q^{-1/2}$ as $\rho \to \infty$. The lower bounds for the first zero should be compared with the bound given in [16, Corollary 4.3]. The upper bound is also given in [15, p. 698].

Proof. We have already proved (i), since $\xi_{k+1} > q^{-1} \xi_k$, (ii) follows from $\xi_1 \geqslant \sigma(q^{-(1/2)\rho})$ with $\sigma(w)$ as in Lemma 3.4, and (iii) follows from (3.2) and Lemma 3.4. ∎

We can sharpen the result by showing that the zeros of $\nu \mapsto w^{-\nu} J_\nu(w; q)$ for fixed $w \in \mathbb{C}$ tend asymptotically to the negative integers for $|\nu|$ large. The method is the same as that of Coulomb [6, Sect. 2], but since Coulomb's paper is rather old we give the details of the proof.

Proposition 3.9. *For each sufficiently small $\varepsilon > 0$ there exists $N > 1$ such that for all integers $m > N$ the function $\nu \mapsto \mathcal{J}_\nu(w; q)$, $w \in \mathbb{C}$, has precisely one simple zero in $\{\nu : |\nu + m| < \varepsilon\}$.*

Proof. Recall that $(q; q)_\infty\, \mathcal{J}_\nu(w; q)$ is analytic in $x = q^\nu$. We compare this function with $(q^{\nu+1}; q)_\infty$, which as an analytic function of $x = q^\nu$

has simple zeros at $x = q^{-m}$, $m = 2, 3, ...$ Now $(q; q)_\infty \mathscr{J}_\nu(w; q) = (q^{\nu+1}; q)_\infty (1 + f_w(x))$ with

$$f_w(x) = \sum_{k=1}^{\infty} \frac{(-1)^k q^{1/2k(k-1)} q^k w^{2k}}{(xq; q)_k (q, q)_k}.$$

For x away from the zeros of $(xq; q)_\infty$ we let δ be defined by $|x - q^{-m}| \geqslant \delta q^{-m}$ for $m = 2, 3, ...,$ so $|(xq; q)_k| \geqslant |1 - xq| \, \delta^{k-1}$ and

$$|1 - xq| \, |f_w(x)| \leqslant \delta \sum_{k=1}^{\infty} \frac{q^{1/2k(k-1)} q^k \, |w|^{2k}}{(q, q)_k \, \delta^k} = \delta\left(\left(-\frac{q\,|w|^2}{\delta}; q\right)_\infty - 1\right)$$

by [8, (1.3.16)]. Observe that the right hand side is a decreasing function of δ independent of x. Hence, given $\delta > 0$ we can find $N = N(\delta) > 1$ such that for $|x| > q^{-N}$ the factor $|1 - xq|$ forces $|f_w(x)| < 1$. Thus $\mathscr{J}_\nu(w; q)$ has no zeros in $|x - q^{-m}| \leqslant \delta q^{-m}$, $m > N$.

Since for $\delta < (q^{-1} - 1)/(1 + q^{-1})$ these disks don't overlap, an application of Rouché's theorem gives that $\mathscr{J}_\nu(w; q)$ has one simple zero in $|x - q^{-m}| \leqslant \delta q^{-m}$, $m > N$. Finally, apply this with $\delta = 1 - q^{\varepsilon}$. ∎

Corollary 3.10. *Let $\rho > 0$. Then for each sufficiently small $\varepsilon > 0$ there exists $N > 1$ such that for $k > N$ we have $q^{1/2+\varepsilon} j_{k+1}(\rho) < j_k(\rho) < q^{1/2} j_{k+1}(\rho)$.*

Corollary 3.10 shows that the zeros $j_k(\rho)$ of the ${}_1\varphi_1$ q-Bessel function $\mathscr{J}_\rho(w; q)$ of order $\rho > 0$ behave like $q^{-(1/2)(k-1)} j_1(\rho)$. So the support of the orthogonality measure in (3.7) behaves like $q^{2-k} j_1(\rho)^2$, $k \to \infty$.

Proof. Take $w = q^{(1/2)\rho}$. Then the zeros of $\nu \mapsto \mathscr{J}_\nu(q^{(1/2)\rho}; q) = \mathscr{J}_\rho(q^{(1/2)\nu}; q)$ correspond to the zeros $j_k(\rho)$ by $q^{(1/2)\nu_k} = j_k(\rho)$, $k \in \mathbb{N}$. By Proposition 3.9 we see that for $\varepsilon > 0$ there is a $N > 1$ such that $\nu_{M+m} \in \{\nu: |\nu + m| < \varepsilon\}$ for some $M \in \mathbb{Z}$ and for all $m > N$. Hence, $\nu_k - \nu_{k+1} < 1 + 2\varepsilon$ or $j_k(\rho) > q^{1/2+\varepsilon} j_{k+1}(\rho)$ for k sufficiently large. ∎

Remark. Since $P_m(w^{-2}x; w; q) = P_m(x; w^{-1}; q)$ we have restricted ourselves to $w \geqslant 1$. The results corresponding to $0 < w \leqslant 1$ can also be obtained directly by this method using the asymptotic properties of $h_{m,\nu}(w; q)$ for $|w| < 1$ given in [16, (3.6)].

Remark. In case $w = 1$ we have only established the indeterminacy of the moment problem, but an explicit orthogonality measure has not been given. Since in the limits like (3.4) and (3.5) the result doesn't show that $w = 1$ is a special point, one might wonder if the limit $\rho \downarrow 0$ in (3.7) does indeed give an orthogonality measure for this case. Indeed, the dependence on ρ in both the ${}_1\varphi_1$ q-Bessel function and its zeros and in the orthogonal polynomials is continuous. But the asymptotics of the orthogonal polynomials seems not easily available. However, the rescaled asymptotics for these polynomials is known, see [21, Theorem 1], in terms of the orthogonality measure for the polynomials as in (2.8), but with $q>1$ so that the recurrence coefficients are bounded.

4. ORTHOGONALITY FOR q-LOMMEL POLYNOMIALS ASSOCIATED WITH JACKSON'S q-BESSEL FUNCTION

In this section we give the analogue of the results of the previous section for the q-Lommel polynomials associated with the Jackson q-Bessel function as introduced by Ismail [9]. The Jackson q-Bessel function is defined by

$$J_\nu^{(2)}(w;q)=\left(\frac{w}{2}\right)^\nu \frac{(q^{\nu+1};q)_\infty}{(q;q)_\infty}\, {}_0\varphi_1\left(\begin{matrix}-\\ q^{\nu+1}\end{matrix};q,-q^{\nu+1}\frac{w^2}{4}\right),$$

so $\tilde{J}_\nu^{(2)}(w;q)=(w/2)^{-\nu}J_\nu^{(2)}(w;q)$ is an entire function in w and in $x=q^\nu$. See Ismail [9] for references to Jackson's papers from 1903–1905. Iteration of the recurrence

$$q^\nu J_{\nu+1}^{(2)}(w;q)=\frac{2}{w}(1-q^\nu)\,J_\nu^{(2)}(w;q)-J_{\nu-1}^{(2)}(w;q) \tag{4.1}$$

leads to the q-Lommel polynomials $h_{m,\nu}^{(2)}(w;q)$ satisfying the recurrence relation

$$h_{m+1,\nu}^{(2)}(w;q)=2w(1-q^{m+\nu})\,h_{m,\nu}^{(2)}(w;q)-q^{n+\nu+1}h_{m-1,\nu}^{(2)}(w;q), \tag{4.2}$$

with $h_{0,\nu}^{(2)}(w;q)=1$, $h_{1,\nu}^{(2)}(w;q)=2w(1-q^\nu)$, see [9].

From (4.2) it follows that the polynomials $h_{m,\nu}^{(2)}(w;q)$ form a set of orthogonal polynomials in w by Favard's theorem, and the orthogonality relations have been determined by Ismail [9]. However, the $h_{m,\nu}^{(2)}(w;q)$ do not form a set of orthogonal polynomials in $x=q^\nu$, but, as observed by Ismail [9], $S_m(q^\nu;w;q)=q^{-(1/2)m\nu}h_{m,\nu}^{(2)}(wq^{(1/2)\nu};q)$ is a polynomial of degree m in $x=q^\nu$ satisfying the three-term recurrence relation

$$S_{m+1}(x;w;q)=2w(1-xq^m)\,S_m(x;w;q)-q^{m-1}\,S_{m-1}(x;w;q) \tag{4.3}$$

with initial conditions $S_0(x;w;q)=1$, $S_1(x;w;q)=2w(1-x)$. So they form a set of orthogonal polynomials for $w\in\mathbb{R}\backslash\{0\}$ by Favard's theorem.

The asymptotic behaviour of the orthogonal polynomials is then given by, see [9, Sect. 3],

$$\begin{aligned} \lim_{m\to\infty} & (2w)^{-m}\, S_m(x; w; q) \\ &= (x; q)_\infty\, {}_0\varphi_1\left(-; x; q, -\frac{1}{4w^2}\right) \\ &= \sum_{k=0}^{\infty} x^k \frac{(-1)^k\, q^{1/2k(k-1)}}{(q; q)_k}\, {}_0\varphi_1\left(-; 0; q, -\frac{q^k}{4w^2}\right) \end{aligned} \tag{4.4}$$

uniformly in x on compact subsets of $\mathbb{C}$. The right hand side is an entire function in x, and also in w^{-1}.

The monic polynomials $s_m(x) = s_m(x; w; q) = (-2w)^{-m}\, q^{-(1/2)\, m(m-1)}\, S_m(x; w; q)$ satisfy

$$x s_m(x) = s_{m+1}(x) + q^{-m} s_m(x) + \frac{q^{-m}}{4w^2}\, s_{m-1}(x), \tag{4.5}$$

with $s_0(x) = 1$, $s_1(x) = x - 1$. The corresponding orthonormal polynomials $u_m(x) = (-\operatorname{sgn} w)^m\, q^{(1/2)\, m} q^{(1/4)\, m(m-1)} S_m(x; w; q) = 2^m\, |w|^m\, q^{(1/2)\, m(m+1)}\, s_m(x)$ satisfy

$$x u_m(x) = \frac{1}{2\,|w|}\, q^{-1/2(m+1)} u_{m+1}(x) + q^{-m} u_m(x) + \frac{1}{2\,|w|}\, q^{-1/2m} u_{m-1}(x), \tag{4.6}$$

with $u_0(x) = 1$, $u_1(x) = 2q^{1/2}\, |w| (x-1)$. Again we may restrict ourselves to $w > 0$, since $S_m(x; -w; q) = (-1)^m\, S_m(x; w; q)$.

LEMMA 4.1. (i) *The moment problem for the polynomials* $S_m(\cdot; w; q)$ *is determinate for* $w > 0$.

(ii) *For the true interval* $[\xi_1, \infty)$ *of orthogonality we have*

(a) $-\infty < \xi_1 < 1$;

(b) $\xi_1 > 0$ *for* $(2w)^{-1} < \sqrt{1-q}$, *and for* $2w > 1 + q^{-1/2}$;

(c) $\xi_1 < 0$ *for* $2w < \sqrt{1+q}$.

Note that the first estimate in (b) is the best for q small and the second is the best for q close to 1.

Proof. From (4.4) we see that $u_m(x) \approx (-2z)^m\, q^{-(1/4)\, m(m-1)} (x; q)_\infty\, {}_0\varphi_1(-; x; q, -(2w)^{-2})$. Take any $x \in \mathbb{C} \backslash \mathbb{R}$ for which the last factor is non-zero. Then we have $\sum_{m=0}^{\infty} |u_m(x)|^2 = +\infty$, so that we find determinacy of the moment problem from [2, Theorem 1.1, p. 503; 19, Theorem 2.9, p. 50].

To prove (ii) we first note that $q^{-m} \to \infty$ as $m \to \infty$ implies that the true interval of orthogonality is of the form $[\xi_1, \infty)$, see [5, p. 109]. Now [5, Theorem 2.1, p. 108] shows that $\xi_1 \geqslant y$, $y \in \mathbb{R}$, if and only if $q^{-m} > y$, $m \in \mathbb{Z}_+$, and $\{\lambda_m(y)\}_{m=1}^{\infty}$ is a chain sequence with

$$\lambda_m(y) = \frac{q^{-m}}{4w^2(q^{-m} - y)(q^{1-m} - y)} = \frac{q^{m-1}}{4w^2(1-q^m y)(1-q^{m-1}y)}.$$

Hence, we can restrict ourselves to $y < 1$.

Using Wall's criterion, see [5, Ex. 5.9, p. 100], we see that $\{aq^{m-1}\}_{m=1}^{\infty}$ is a chain sequence if $a < 1-q$. On the other hand, the minimal parameter sequence, see [5, Chap. III], starts with $g_0 = 0$, $g_1 = a$, $g_2 = aq/(1-a)$. From the condition $0 < g_1 < 1$ we conclude that for $a \leqslant 0$ or $a \geqslant 1$ the sequence $\{aq^{m-1}\}_{m=1}^{\infty}$ is not a chain sequence, and from $0 < g_2 < 1$ we conclude that for $a \geqslant (1+q)^{-1}$ it is not a chain sequence.

For $0 \leqslant y < 1$ we get $(1-q^m y)(1-q^{m-1}y) \geqslant (1-y)(1-qy)$ for $m \in \mathbb{N}$, hence $\lambda_m(y) \leqslant aq^{m-1}$ with $a^{-1} = 4w^2(1-y)(1-qy)$. By Wall's comparison test, see [5, Theorem 5.7, p. 97], it follows that $\lambda_m(y)$ forms a chain sequence if aq^{m-1} forms one. If $(2w)^{-1} < \sqrt{1-q}$ we can find $y > 0$ small such that $4w^2(1-y)(1-qy) > (1-q)$ implying $\lambda_m(y)$ forming a chain sequence. Hence, the first part of (b) follows. If we now put $y = 0$, then (c) follows, since $\lambda_m(0)$ doesn't form a chain sequence for $2w < \sqrt{1+q}$.

To prove that ξ_1 is bounded from below we can use Gershgorin's theorem as in the proof of Lemma 3.4. We see that all zeros of the orthonormal polynomials $u_m(x)$ are contained in $[-b, \infty)$ for $b \in \mathbb{R}$ such that

$$q^{-m} - q^{-1/2m}(2w)^{-1}(1+q^{-1/2}) + b \geqslant 0, \qquad m \in \mathbb{Z}_+.$$

Then $-b \leqslant \xi_1$, and it suffices to take $b \geqslant (1+q^{-1/2})^2/(16w^2)$. This proves (a). If we take $b = 0$ we see that for $2w \geqslant 1 + q^{-1/2}$ we have $q^{-m} - q^{-(1/2)m}(2w)^{-1}(1+q^{-1/2}) \geqslant \tau(w)$, $m \in \mathbb{Z}_+$, with $\tau(w) = 1 - (1+q^{-1/2})/(2w) \geqslant 0$, so that $\xi_1 \geqslant \tau(w)$ in this case and the latter part of (b) follows. ∎

Let again ξ_i denote the limit of the ith zero of $S_m(\cdot; w; q)$ as $m \to \infty$. Then it follows that the orthogonality measure is supported on $\Xi = \{\xi_1, \xi_2, ...\}$, see [5, Chap. IV].

To determine the orthogonality measure we first consider the associated monic polynomials $s_m^{(1)}(x) = s_m^{(1)}(x; z; q)$ defined by

$$xs_m^{(1)}(x) = s_{m+1}^{(1)}(x) + q^{-m-1}s_m^{(1)}(x) + \frac{q^{-m-1}}{4w^2}\, s_{m-1}^{(1)}(x), \tag{4.7}$$

with $s_0(x)=1$, $s_1(x)=x-q^{-1}$. Multiply by q, and take the new variable $y=qx$ to see that $s_m^{(1)}(x;w;q)=q^{-m}s_m(qx;wq^{-1/2};q)$. Hence, the Stieltjes transform $L(x;w;q)$ of the orthogonality measure is given by

$$L(x;w;q)=\lim_{m\to\infty}\frac{s_{m-1}^{(1)}(x)}{s_m(x)}=-\frac{(xq;q)_\infty\,{}_0\varphi_1(-;qx;q,q/(4w^2))}{(x;q)_\infty\,{}_0\varphi_1(-;x;q,1/(4w^2))} \tag{4.8}$$

by (4.4).

Lemma 4.2. *The denominator and numerator of L in* (4.8) *have no common zeros.*

Proof. From

$$(1-x)(xq;q)_\infty\,{}_0\varphi_1\left(-;xq;q,-\frac{q}{4w^2}\right)$$
$$-\frac{1}{4w^2}(xq^2;q)_\infty\,{}_0\varphi_1\left(-;xq^2;q,-\frac{q^2}{4w^2}\right)$$
$$=(x;q)_\infty\,{}_0\varphi_1\left(-;x;q,-\frac{1}{4w^2}\right),$$

cf. [9, Sect. 2], it follows that if $x\neq 0$ is a common zero of the numerator and denominator, then

$$(xq^k;q)_\infty\,{}_0\varphi_1\left(-;xq^k;q,-\frac{q^k}{4w^2}\right)=0,\qquad k\in\mathbb{Z}_+,$$

implying that the entire function $z\mapsto(z;q)_\infty\,{}_0\varphi_1(-;z;q,-z/(4w^2x))$ is identically zero, which is absurd.

By (4.4) we see that

$$L(0;w;q)=-\frac{{}_0\varphi_1(-;0;q,q/(4w^2))}{{}_0\varphi_1(-;0;q,1/(4w^2))}.$$

If both numerator and denominator are zero then also ${}_0\varphi_1(-;0;q,q^k/(4w^2))=0$ for $k\in\mathbb{Z}_+$ as follows from the second order q-difference equation, cf. [8, Ex. 1.13], giving a contradiction in a similar way. ∎

Lemma 4.3. *For $w>0$ the function $L(x;w;q)$ is meromorphic and it has a Mittag–Leffler expansion*

$$L(x;w;q)=\sum_{k=1}^{\infty}\frac{B_k}{x-\xi_k}$$

with $B_k > 0$, $\sum_{k=1}^{\infty} B_k = 1$, *and* $-\infty < \xi_1 < \xi_2 < \cdots$ *tending to infinity. Moreover, the orthogonality measure for the orthogonal polynomials* $S_m(\cdot; w; q)$ *is uniquely determined and given by a purely discrete measure with masses* B_k *at the points* ξ_k.

Proof. As in the proof of Lemma 3.5, we use Maki [18, Theorem 4.9], which holds for all $w > 0$ since $\limsup_{m\to\infty} q^{-m}/(4w^2 q^{1-2m}) = 0 < 1/4$ and $q^{-m} \to \infty$. Then $\xi_k \to \infty$, since the support of the orthogonality measure has to be infinite. ∎

THEOREM 4.4. *Let* $w > 0$.

(i) *The zeros of* $x \mapsto (x; q)_\infty\, {}_0\varphi_1(-; x; q, 1/(4w^2))$ *are real, simple, and form a denumerable set* $-\infty < \xi_1 < \xi_2 < \cdots$ *with* $\xi_k \to \infty$ *as* $k \to \infty$. *Moreover, the zeros of* $x \mapsto (x; q)_\infty\, {}_0\varphi_1(-; x; q, 1/(4w^2))$ *interlace with those of* $x \mapsto (xq; q)_\infty\, {}_0\varphi_1(-; qx; q, q/(4w^2))$.

(ii) *The polynomials* $S_m(\cdot; w; q)$ *satisfy the orthogonality relations*

$$-\sum_{k=1}^{\infty} S_m(\xi_k; w; q)\, S_n(\xi_k; w; q)\, \frac{(\xi_k q; q)_\infty\, {}_0\varphi_1(-; q\xi_k; q, q/(4w^2))}{(\partial/\partial x)((x; q)_\infty\, {}_0\varphi_1(-; x; q, 1/(4w^2)))|_{x=\xi_k}}$$
$$= \delta_{nm} q^{-m} q^{-1/2m(m-1)}.$$

All masses are positive, and the measure is uniquely determined.

Proof. The proof is a slight adaptation of the proof of Theorem 3.6. ∎

In case $\xi_1 > 0$ we put $x = q^\nu$, $\xi_k = q^{\mu_k}$. Then we can rewrite Theorem 4.4 in terms of the Jackson q-Bessel function as follows.

COROLLARY 4.5. *Let* $2w > min(1 + q^{-1/2}, 1/\sqrt{1-q})$.

(i) *The zeros of* $\nu \mapsto \mathscr{J}_{\nu-1}^{(2)}(w^{-1} q^{-(1/2)\nu}; q)$ *are real, simple, and form a denumerable set* $\cdots < \mu_{k+1} < \mu_k < \cdots < \mu_1 < \infty$ *with* $\mu_k \to -\infty$ *as* $k \to \infty$. *Moreover, the zeros of* $\nu \mapsto \mathscr{J}_{\nu-1}^{(2)}(w^{-1} q^{-(1/2)\nu}; q)$ *interlace with the zeros of* $\nu \mapsto \mathscr{J}_{\nu}^{(2)}(w^{-1} q^{-(1/2)\nu}; q)$.

(ii) *The q-Lommel polynomials defined by* (4.2) *satisfy the orthogonality relations*

$$-2\ln q \sum_{k=1}^{\infty} q^{-1/2m\mu_k} h_{m,\mu_k}^{(2)}(wq^{1/2\mu_k}; q)\, q^{-1/2n\mu_k} h_{n,\mu_k}^{(2)}(wq^{1/2\mu_k}; q)\, q^{\mu_k}$$
$$\times \frac{\mathscr{J}_{\mu_k}^{(2)}(w^{-1} q^{-1/2\mu_k}; q)}{(\partial/\partial\nu)(\mathscr{J}_{\nu-1}^{(2)}(w^{-1} q^{-1/2\nu}; q))|_{\nu=\mu_k}} = \delta_{nm} q^{-m} q^{-1/2m(m-1)}.$$

The analogue of Coulomb's result, Proposition 3.9, is also valid for Jackson's q-Bessel function, with only minor changes in the estimate involved.

PROPOSITION 4.6. *For each sufficiently small* $\varepsilon > 0$ *there exists* $N > 1$ *such that for all integers* $m > N$ *the function* $\nu \mapsto \mathcal{J}_\nu^{(2)}(wq^{-(1/2)\nu}; q)$, $w \in \mathbb{C}$, *has precisely one simple zero in* $\{\nu \colon |\nu + m| < \varepsilon\}$.

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